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Derivation of conditionally exactly solvable potentials from quasi-exactly solvable ones and determination of its bound states

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Abstract A connection between conditionally exactly solvable potentials and certain quasi-exactly solvable potentials are established when we attempt to determine the exact bound states of the confining potential $ar^{2/3} + br^{2/3}$ In this case, the angular momentum 'l' is found to assume a fractional value For integer 'l' one requires the addition of a centrifugal term d / r^2 to the former giving rise to the recently introduced conditionally exactly solvable potential with a specific value of d. It is further shown that 'd' can have a number of values depending on the value of 'l' and hence an infinite number of potentials can be found with same eigen value and eigen function

Keywords . Quasi exactly solvable potentials, bound states, angular momentum

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1. Introduction

From the early days of quantum mechanics, enormous efforts have been devoted to search for the exact solutions of time independent Schrödinger equation for various potentials of physical interest. This is mainly due to the importance of such solutions in many branches of Physics and due to the applications of these solutions as basis to perform perturbative calculations. However, although only a few potentials are exactly solvable for its entire spectrum, a large number of potentials can be solved exactly for a finite number of eigenstates provided its parameters satisfy certain constraining relations between themselves. These new class of potentials which represents the actual interaction in relevant physical problem or approximates it, is known as quasi exactly solvable (QES) potentials [1-5]. Very recently Dutra [6] has introduced another type of potential known as the conditionally exactly solvable (CES) potentials. They modify the usual potential in quantum mechanics in a specific way such that they are exactly solvable when a parameter of the potential assumes a specific value. These potentials are positioned midway between the exactly solvable potentials and quasi exactly solvable ones. Their principal feature is that one can find the entire S-Wave Spectrum of a given potential, provided that some of the parameters are conveniently fixed. CES potentials have currently been discussed by many authors in different contexts [7-9]. Grosche has analysed CES potentials in detail using path integral formalism [10]. Dutt *et al.* [11] have recently mentioned a new class of CES potentials. We, in the present work, try to determine the exact bound states of the Schrödinger equation for a strongly anharmonic, symmetrically fractional power potential of the form $V(r) = ar^{2/3} + br^{-2/3}$. This potential has been suggested as a candidate for the interaction between constituent quarks inside the light as well as heavy mesons [12]. In addition, this potential added with a centrifugal term d/r^2 constitutes one of the few conditionally exactly solvable problems introduced by Dutra [6]. Our work is thus motivated by the twin goals of obtaining the bound states of the potential in a very simple and elegant way by assuming an *ansatz* of the form exp [g(r)] not attempted earlier and of providing some insight into the transition of this problem to a CES problem by pointing out the physical necessity of adding the centrifugal term in a particular way. It is further pointed out that the particular value of the coefficient d of the additional centrifugal term in the work of Dutra [6] corresponds to S-wave spectrum *i.e.* for $l = 0$. Other specific values for this coefficient corresponding to higher values of the angular momentum 'l' are found out in the present work, *i.e.* we show that the coefficient 'd' is a function of the angular momentum 'l'.

2. Theory

In dimensionless variables ($\hbar = 2m = 1$) the Schrödinger equation for the reduced function $\phi(r)$ [$\frac{\phi(r)}{r} = \psi(r)$, with $\psi(r)$ being the Schrödinger wave function] for the potential $V(r)$ reads

$$\phi''(r) - \left[-E + V(r) + \frac{l(l+1)}{r^2}\right]\phi(r) = 0 \tag{1}$$

with $V(r)$ given by

$$V(r) = ar^{2/3} + br^{-2/3}. \tag{2}$$

The *ansatz* $\phi(r) = \exp\left[\alpha r^{4/3} + \beta r^{2/3} + \delta\, lnr\right]$ (3)

leads to the following second order differential equation in $\phi(r)$:

$$\phi''(r) - \left[\frac{16\alpha\beta}{9} + \frac{16\alpha^2}{9} r^{2/3} + \left(\frac{4\alpha}{9} + \frac{4}{9}\beta^2 + \frac{8}{3}\alpha\delta\right)r^{-2/3}\right.$$

$$\left. + \left(\frac{4}{3}\beta\delta - \frac{2}{9}\beta\right)r^{-4/3} + \frac{\delta(\delta-1)}{r^2}\right]\phi(r) = 0. \tag{4}$$

Comparison of the above equation with that in (1) with $V(r)$ given by (2) leads to the following set of relations

$$E = -\frac{16}{9}\alpha\beta, \tag{5a}$$

$$\alpha = \pm\frac{3}{4}\sqrt{a}. \tag{5b}$$

Negative sign of α is chosen to satisfy physical condition

$$\left[\frac{4}{9}\beta^2+\frac{4}{9}\alpha+\frac{8}{3}\alpha\delta\right]=b, \tag{5c}$$

$$\left[\frac{4}{3}\beta\delta-\frac{2}{9}\beta\right]=0 \tag{5d}$$

$$\delta(\delta-1)=l(l+1) \tag{5e}$$

From (5*e*), δ = either $-l$ or $(l+1)$ However, $\psi(r)$ should remain finite as $r\to 0$. So we have

$$\delta=(l+1) \tag{6}$$

Again from (5d) $\delta=\frac{1}{6}$

which in turn gives from (6)

$l=-\frac{5}{6}$. This makes the centrifugal term in eq. (1) as

$$\frac{l(l+1)}{r^2}=-\frac{5}{36}\cdot\frac{1}{r^2}$$

This negative fractional value of l in 3-dimensional problem is unphysical and undesirable as the angular momentum l is required to be integers 0, 1, 2, ... *etc*. However, if a term $\frac{d}{r^2}$ with $d=-\frac{5}{36}$ is added to the potential $V(r)$ in (2), l assumes the value 0 with eq. (5e) changing to $\delta(\delta-1)=d+l(l+1)$.

But $$V(r)=ar^{2/3}+br^{-2/3}+\frac{d}{r^2} \tag{7}$$

with $d=-\frac{5}{36}$ is one of the recently introduced conditionally exactly solvable potentials [6, 10]. Thus, we see the transition of the quasi-exactly solvable potential (2) into the conditionally exactly solvable potential as expressed in (7). This also provides one physical explanation for choosing 'd' in a particular way in the previous work on conditionally exactly solvable problems [6, 10]. Previously, the inclusion of the additional centrifugal term was explained either as a necessity for the existence of exact solution of the problem [6, 7] or on the basis of existence of a supersymmetric structure of the potential [9]. It is further noted that corresponding to the values of l = 1, 2, 3 ..... l, the centrifugal terms to be added to the potential (2) are respectively given by

$$\frac{d}{r^2}=-\frac{77}{36r^2},-\frac{221}{36r^2},-\frac{437}{36r^2},\ldots\left[\frac{5}{36r^2}\quad l(l+1)\right.$$

This makes the net centrifugal term $\left[\frac{d+l(l+1)}{r^2}\right]$ in eq. (1) equal to $-\frac{5}{36r^2}$ corresponding to any integer value of 1. This generalisation of the coefficient 'd' corresponding to higher values of the angular momentum 'l' is technically straightforward. It shows that instead of obtaining the entire spectrum of a particular potential for S-wave only one gets a sequence of potentials with specified values 'd' each of which corresponds to an integer value of the angular momentum 'l' and has same eigenvalue and eigenfunction.

Substitution of the value of $\delta = \frac{1}{6}$ in eq. (5c) yields

$$\beta = \frac{3}{2}\left[b + \frac{2}{3}\sqrt{a}\right]^{1/2}.$$

Thus the eigen energy and the eigenfunction for the or CES in eq. (7) are obtained as

$$E = \frac{16\alpha\beta}{9} = 2\sqrt{a}\left[b + \frac{2}{3}\sqrt{a}\right]^{1/2} \tag{8a}$$

$$\phi(r) = r^{1/6} exp\left[-\frac{3}{4}\sqrt{ar^{4/3}} \pm \frac{3}{2}\left(b + \frac{2}{3}\sqrt{a}\right)^{1/2} r^{2/3}\right] \tag{8b}$$

3. Generalisation to a class of solutions

We generalise the *ansatz* (3) to

$$\phi(r) = exp\left[\alpha r^{4/3} + \beta r^{2/3} + ln\sum_{n=0}^{\infty}\omega_n r^{\frac{2n}{3}+\delta}\right]. \tag{9}$$

Substituting this in eq. (1) for the potential (2) and setting the coefficients of the term $r^{(\frac{2n}{3}+\delta)}$ equal to zero in the resultant equation, leads to the following recurrence relation for the coefficients ω_n:

$$A_n\omega_n + B_{n+1}\omega_{n+1} + C_{n+2}\omega_{n+2} = 0 \tag{10}$$

with

$$A_n = \frac{4}{9}\beta^2 + \frac{8}{3}\alpha\left(\frac{2n}{3} + \delta\right) + \frac{4\alpha}{9} - b, \tag{11a}$$

$$B_n = \frac{2}{9}\beta + \frac{4}{3}\beta\left(\frac{2n}{3} + \delta\right), \tag{11b}$$

$$C_n = \left(\frac{2n}{3} + \delta\right)\left(\frac{2n}{3} + \delta - 1\right) - l(l+1). \tag{11c}$$

We have also $-16\alpha\beta = E$, $\frac{16\alpha^2}{9} = a$ as in (5a, b).

Now ω_0 being the first nonvanishing coefficient

$C_0 = 0$

or $\delta(\delta-1)=l(l+1)$

and we take $\delta=(l+1)$ as before.

For a polynomial solution with a finite

$$n = k$$

$$\omega_k \neq 0 \text{ and } \omega_{k+1} = \omega_{k+2} \cdots\cdots\cdots = 0$$

This gives $A_k = 0$. From (11a)

$$\beta = \pm\frac{3}{2}\left[b+\sqrt{a}\left(\frac{4n}{3}+2l+\frac{7}{3}\right)\right]^{1/2} \tag{12}$$

and the eigen energy follows

$$E = -\frac{16}{9}\alpha\beta = \pm 2\sqrt{a}\left[b+\sqrt{a}\left(\frac{4n}{3}+2l+\frac{7}{3}\right)\right]^{1/2}. \tag{13}$$

Again the condition $\omega_{k+1}=0$ for the existence of non trivial solution of (10) is equivalent to the vanishing of the $(k+1)$ dimensional determinant

i.e.

$$D_{k+1}(a,b,c,E) = \begin{vmatrix} B_0 & C_1 & 0 & & \\ A_0 & B_1 & C_2 & & \\ & & & & \\ & & A_{k-2} & B_{k-1} & \\ & & & A_{k-1} & B_k \end{vmatrix} = 0 \tag{14}$$

Eq. (14) provides the constraints to be satisfied by the parameters of the quasi potential (2). From the constraining relation (14) when K equals Zero, we have

$$B_0 = 0$$

or $\delta = \frac{1}{6}$ as before and $l = -\frac{5}{6}$.

Thus as pointed out before, CES potential $\left(ar^{2/3}+br^{-2/3}+\frac{d}{r^2}\right)$ results on the inclusion of an additional term $\frac{d}{r^2}$ with $d=-\frac{5}{36},-\frac{77}{36},-\frac{221}{36}$ *etc.* corresponding to $l=0,1,2,\ldots\ldots$ respectively to the potential (2).

Finally, the eigen energy and eigenfunction in this case corresponding to $\delta=\frac{1}{6}$ can be expressed from (11a or 13) as

$$\beta = \frac{2}{3}\left[b\mp\frac{2}{3}\sqrt{a}(2n+1)\right]^{1/2} \text{ leading to}$$

$$E = -2\sqrt{a}\left[b\mp\frac{2}{3}\sqrt{a}(2n+1)\right]^{1/2}. \tag{15a}$$

This expression for the energy is in agreement with that in the work of Dutra (6) and the nature of this expression and the condition for its boundedness are the same as discussed in their work.

$$\phi(r) = r^{1/6} exp\left[\pm\frac{3}{4}\sqrt{a}\, r^{4/3} \pm \frac{3}{2}\left[b \pm \frac{2\sqrt{a}}{3}(2n+1)\right] r^{2/3} + ln\sum_{n=0}^{\infty}\omega_n r^{2n/3}\right]. \quad (15b)$$

4. Concluding remarks

In our work, we have discussed the transition of a QES potential into the CES potential $ar^{2/3} + br^{-2/3} - \frac{5}{36r^2}$ pointing out the physical necessity of adding the centrifugal term with specific d. It is further shown that instead of one particular value $d = -\frac{5}{36}$ it can assume a number of values depending on the value of l. Also the eigen energy and the eigen function of the CES potential in eq (7) as well as the confining potential $ar^{2/3} + br^{-2/3}$ are determined in this work in a very simple and elegant method.

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